Periodic Solution of Nonmonotonic Predator-prey System with Periodic Coefficients and Undercrowding Effect

Yan Yan[1], Zhanji Gui[2,a*], and Kaihua Wang[3]

[1] Networking Academy, Haikou college of Economics, Haikou, Hainan, 571158, P.R.China

[2] Department of Software Engineering, Hainan College of Software Technology, Qionghai, Hainan, 571400, P.R. China

[3] School of Mathematics and Statistics, Hainan Normal University, Haikou, Hainan, 571158, P.R. China

[*a] 624163666@qq.com

*The corresponding author

Keywords: Periodic Solution; Degree theorem; Undercrowding effect.

Abstract: The paper studies nonmonotonic predator and prey model with periodic coefficients and undercrowding effect. Based on some mathematical analysis theories, the paper makes use of Mawhin's continuation theorem, which gives the existence theorem of periodic solution of the sufficient conditions. In the end, the paper gives a numerical simulation example, which gets the validity of the results.

1. Introduction

As is well known, a framework of predator-prey interaction in this wide sense was established by Volterra [1] in a set of simultaneous differential equations having the form

$$\begin{cases} \dot{x}(t) = f(x) - g(x, y), \\ \dot{y}(t) = u\big(g(x, y), y\big) - v(t), \end{cases} \tag{1}$$

where x represent densities of predator and y represent densities of prey. The functions f is density of the rates of prey reproduction, g is density of prey death due to predation, u is predator reproduction, and v is predator death,which are given concrete forms, $f(x) = ax^2/(N_1 + x) - bx - cx^2$, $g(x, y) = mn \cdot xy/(m + nx)$, $u\big(g(x, y), y\big) = g(x, y) \cdot y \cdot d/(N_2 + y)$, $v(y) = ey.$ Thus, incorporating this concrete forms into the framework of Eq.1, we have the basic differential equation model of predator-prey interaction[2],

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{ax^2}{N_1 + x} - bx - cx^2 - \dfrac{mnxy}{m + nx}, \\ \dfrac{dy}{dt} = \dfrac{dmnxy^2}{(m + nx)(N_2 + y)} - ey. \end{cases}$$

In this section, we will account the corresponding predator and prey model with periodic undercrowding effect and coefficients , we obtain

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{a(t)x^2}{N_1(t) + x} - b(t)x - c(t)x^2 - \dfrac{m(t)n(t)xy}{m(t) + n(t)x}, \\ \dfrac{dy}{dt} = \dfrac{d(t)m(t)n(t)xy^2}{(m(t) + n(t)x)(N_2(t) + y)} - e(t)y, \end{cases} \tag{2}$$

where x represent densities of predator and y represent densities of prey.

  DOI: 10.25236/mmmce.2019.119

$a(t), b(t), c(t), d(t), e(t),\ n(t), m(t), N_1(t)$ and $N_2(t)$ are strictly positive periodic functions , at the same time continuous and bounded with $\omega > 0$.

2. Existence of Periodic Solutions

In this section, in order to explore the existence of periodic solution of Eq. 2, we use Mawhin's continuation theorem to prove the existence of periodic solutions of system (1). More details can be referred to [3].

Lemma 1.([3]) Let's define X and Y are Banach spaces. Definer an operator equation $Lx = \lambda Nx$, and $L : DomL \bigcap X \to Y$ is a Fredholm operator of index zero , $\lambda \in [0,1]$ is a parameter, then there exist two projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = KerL$ and $\operatorname{Im} L = KerQ$. Assume that $N : \overline{\Omega} \to Y$ is L- compact on $\overline{\Omega}$, where Ω is open bounded in X. Furthermore, assume that

(a)every solution $x \in \partial\Omega \bigcap DomL$ for $\lambda \in (0,1)$, $Lx \neq \lambda Nx$;

(b) QNx is not zero for each $x \in \partial\Omega \bigcap KerL$;

(c) the *brouwer degree* $\deg\{JQN, \Omega \bigcap KerL, 0\}$ is not equal to zero.

Then in $\overline{\Omega} \bigcap DomL$, $Lx = Nx$ has at least one solution .

First, let's make some preparations.

$$\Gamma_1(t) = \frac{a(t) - c(t)N_1(t)}{N_1(t)}, \Gamma_2(t) = \frac{m(t)n(t)}{m(t) + n(t)\exp\{H_1\}}, \Gamma_3(t) = \frac{d(t)m(t)}{N_2(t)},$$

$$\overline{u} = \frac{1}{\omega}\int_0^\omega u(s)ds,\ H_1 = \ln\left(\frac{\overline{a} - \overline{b}}{\overline{c}}\right) + 2\overline{a}\omega.$$

Theorem 1. If $\ln\dfrac{\overline{b}}{\overline{\Gamma}_1} \geq 2\overline{a}\omega,\ \ln\dfrac{\overline{e}}{\overline{\Gamma}_3} \geq 2\overline{e}\omega$ so that there is at least one positive ω-periodic solution for Eq.2.

Proof. We can let $x = \exp\{u_1\},\ y = \exp\{u_2\},$ then the Eq. 2 becomes

$$\begin{cases} \dot{u}_1(t) = \left(\dfrac{a(t)}{N_1(t) + \exp\{u_1\}} - c(t) - \dfrac{m(t)n(t)\exp\{u_2\}}{m(t) + n(t)\exp\{u_1\}}\right)\exp\{u_1\} - b(t), \\ \dot{u}_2(t) = \dfrac{d(t)m(t)\exp\{u_1 + u_2\}n(t)}{[m(t) + n(t)\exp\{u_1\}][N_2(t) + \exp\{u_2\}]} - e(t). \end{cases} \tag{3}$$

In order to use Lemma 1, we set $X = Z = \{u = (u_1(t), u_2(t) \in C(R, R^2) \mid u(t+\omega) = u(t)\}$, then it is standard to show that both X and Z are Banach space when they are endowed with the norm

$\| u \| = \|(u_1(t), u_2(t))\| = \max\limits_{t\in[0,\omega]} | u_1(t) | + \max\limits_{t\in[0,\omega]} | u_2(t) |$.

Set $L : \mathrm{Dom}\, L \subset X \to Z$ as $Lu = u'$ and $Pu = Qu = \overline{u}$.

L is a Fredholm operator of index zero that can be easily proved, N is L- compact on $\overline{\Omega}$ for any given open and P, Q are mappings, Ω is bound subset in X .

According to equation $Lu = \lambda Nu$, assume that $u(t) = (u_1, u_2) \in Z$ is a solution of Eq. 4 for a certain $\lambda \in (0,1)$. According to Eq. 3 over the interval $[0, \omega]$, we can

$$\int_0^\omega \frac{a(t)\exp\{u_1\}}{N_1(t) + \exp\{u_1\}}dt = \int_0^\omega \left[b(t) + c(t)\exp\{u_1\} + \frac{m(t)n(t)\exp\{u_2\}}{m(t) + n(t)\exp\{u_1\}}\right]dt, \tag{4}$$

$$\bar{e}\,\omega = \int_0^{\omega} \frac{d(t)m(t)n(t)\exp\{u_1+u_2\}}{[m(t)+n(t)\exp\{u_1\}][N_2(t)+\exp\{u_2\}]}dt. \tag{5}$$

From Eq. 3-Eq. 5, we obtain

$$\begin{aligned}\int_0^{\omega}|u_1'|dt &\le \lambda\int_0^{\omega}\left[\frac{a(t)\exp\{u_1\}}{N_1(t)+\exp\{u_1\}}+b(t)+c(t)\exp\{u_1\}+\frac{m(t)n(t)\exp\{u_2\}}{m(t)+n(t)\exp\{u_1\}}\right]dt \\ &= 2\int_0^{\omega}\frac{a(t)\exp\{u_1\}}{N_1(t)+\exp\{u_1\}}dt \le 2\bar{a}\,\omega,\end{aligned} \tag{6}$$

$$\int_0^{\omega}|u_2'|dt \le \lambda\int_0^{\omega}\left[\frac{d(t)m(t)n(t)\exp\{u_1+u_2\}}{[m(t)+n(t)\exp\{u_1\}][N_2(t)+\exp\{u_2\}]}+e(t)\right]dt \le 2\int_0^{\omega}e(t)dt = 2\bar{e}\,\omega. \tag{7}$$

Note that $(u_1(t),u_2(t))\in Z$, then exists $\xi_i,\eta_i\in[0,\omega]$,such that $u_i(\xi_i)=\inf\limits_{t\in[0,\omega]}u_i(t)$, $u_i(\eta_i)=\sup\limits_{t\in[0,\omega]}u_i(t)$. By Eq. 4-Eq. 6, we have

$$\bar{a}\,\omega = \int_0^{\omega}a(t)dt \ge \int_0^{\omega}\frac{a(t)\exp\{u_1\}}{N_1(t)+\exp\{u_1\}}dt \ge \int_0^{\omega}\big(b(t)+c(t)\exp\{u_1(\xi_1)\}\big)dt \ge \bar{b}\,\omega+\bar{c}\,\omega\exp\{u_1(\xi_1)\},$$

which implies

$$u_1(\xi_1) \le \ln\frac{\bar{a}-\bar{b}}{\bar{c}}, \qquad u_1(t) \le 2\bar{a}\,\omega+\ln\left(\frac{\bar{a}-\bar{b}}{\bar{c}}\right) =: H_1. \tag{8}$$

By Eq. 4, we also have

$$\begin{aligned}&\int_0^{\omega}\frac{a(t)}{N_1(t)}\exp\{u_1\}dt \ge \int_0^{\omega}\frac{a(t)}{N_1(t)+\exp\{u_1\}}\exp\{u_1\}dt \ge \int_0^{\omega}\big(b(t)+c(t)\exp\{u_1(t)\}\big)dt \\ &\exp\{u_1(\eta_1)\}\int_0^{\omega}\left(\frac{a(t)}{N_1(t)}-c(t)\right)dt \ge \int_0^{\omega}b(t)dt;\end{aligned}$$

which implies

$$u_1(\eta_1) \ge \ln\frac{\bar{b}}{\bar{\Gamma}_1}, \qquad u_1(t) \ge \ln\frac{\bar{b}}{\bar{\Gamma}_1}-2\bar{a}\,\omega =: L_1. \tag{9}$$

By Eq. 4, we also have

$$\begin{aligned}\bar{a}\,\omega &= \int_0^{\omega}a(t)dt \ge \int_0^{\omega}\frac{a(t)}{N_1(t)+\exp\{u_1\}}\exp\{u_1\}dt \\ &\ge \int_0^{\omega}b(t)dt+\int_0^{\omega}\frac{m(t)n(t)\exp\{u_2\}}{m(t)+n(t)\exp\{u_1\}}dt \ge \bar{b}\,\omega+\bar{\Gamma}_2\omega\exp\{u_2(\xi_2)\},\end{aligned}$$

which implies

$$u_2(\xi_2) \le \ln\frac{\bar{a}-\bar{b}}{\Gamma_2}, \quad u_2(t) \le u_2(\xi_2)+\int_0^{\omega}|u_2'(t)|dt \le \ln\frac{\bar{a}-\bar{b}}{\bar{\Gamma}_2}+2\bar{e}\,\omega =: H_2. \tag{10}$$

By Eq. 5, we also have

$$\int_0^{\omega}\frac{d(t)m(t)\exp\{u_2\}}{N_2(t)}dt \ge \int_0^{\omega}\frac{d(t)m(t)n(t)\exp\{u_1+u_2\}}{n(t)N_2(t)\exp\{u_1\}}dt$$
$$\ge \int_0^{\omega}\frac{d(t)m(t)n(t)\exp\{u_1+u_2\}}{[m(t)+n(t)\exp\{u_1\}][N_2(t)+\exp\{u_2\}]}dt \ge \bar{e}\,\omega \,.$$

which implies

$$y_1(\eta_1)\ge \ln\frac{\bar{e}}{\bar{\Gamma}_3},\quad u_2(t)\ge u_2(\eta_2)-\int_0^{\omega}|u_2'(t)|dt=\ln\frac{\bar{e}}{\bar{\Gamma}_3}-2\bar{e}\,\omega=:L_2. \tag{11}$$

Hence, Eq. 8-Eq.10 and Eq.11 imply that $\sup_{t\in[0,\omega]}|u_1(t)|<\sup_{t\in[0,\omega]}\{|L_1|,|H_1|\}\;:=D_1$, $\sup_{t\in[0,\omega]}|u_2(t)|\;<\sup_{t\in[0,\omega]}\{|L_2|,|H_2|\}:=D_2$. Clearly, D_i ($i=1,2$) are independent of λ. Denote $D=D_1+D_2+D_3$, where $D_3>0$ is taken sufficiently large, such that $\max\{|u_1|,|u_2|\}\le D_3$ and define $\Omega=\{u(t)\in Z:\|u\|<D\}$. By now we have proved that Ω satisfies all the requirements of Lemma 1. Hence, we derive that system (1) has at least one positive ω- periodic solution. The proof is complete.

3. An Example of Numerical Simulations

In this section, we shall discuss an example to illustrate main results. For system (2), we take

$a(t)=6+3\cos t,\quad b(t)=1/5+0.2\cos t,\quad c(t)=0.4-0.2\sin t,\quad n(t)=0.8+0.3\cos t,$
$N_1(t)=1-0.2\cos t,\quad N_2(t)=0.23-0.23\sin t,\quad m(t)=0.9-0.3\sin t,\quad d(t)=0.4+0.2\cos t,$
$e(t)=0.06+0.06\cos t.$

Obviously, this condition satisfies theorem 1, Thus Eq. 2 has a unique 2π-periodic solution (See Fig. 1-Fig.3, where $(x(0),y(0))^T=(0.1,0.1)^T$.

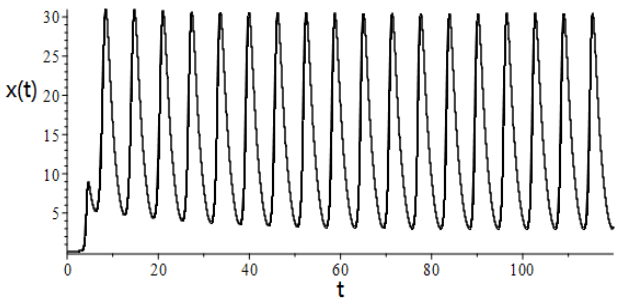

Figure. 1 $x(t)$ picture at time t of system (2)

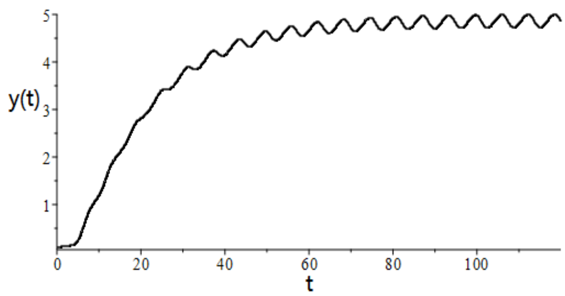

Figure. 2 $y(t)$ picture at time t of system (2) of system (2)

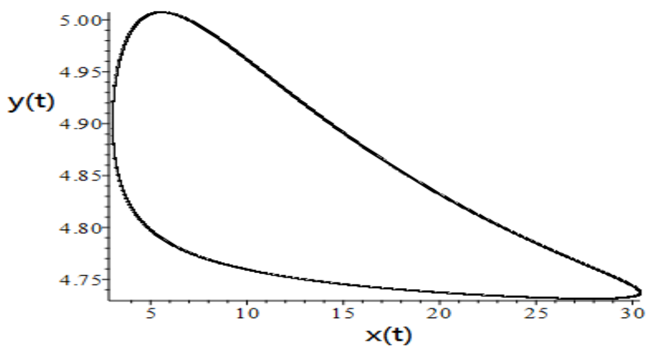

Figure. 3 Phase portrait of 2π -periodic solution of system (2)

Acknowledgments

This work is supported by the Hainan Nature Science Foundation (117164) in China.

References

[1] V. Volterra, Variazioni e fluttuazioni dei numero d'individui in specie animali conviventi. Mem. Acad. Lincei. 2(1926)31-113. (English translation in: Chapman, R. N. (1931). "Animal Ecology" McGraw-Hill, New York.)

[2] E. Kuno, Principles of Predator-Prey interaction in theoretical experimental and natural population systems. *Advances in Ecological Research* 16 (1987)250-255

[3] R. E. Gaines, J. L. Mawhin, Coincidence degree and nonlinear differential equations. Springer-Verlag, Berlin, 1977.